FUNCTIONS AND SERIES APPROXIMATION

Reference :

James G., Modern Engineering Mathematics.

The material is scattered around in Chapters 1, 6 and 8.
Ch. 1.4 for hyperbolic functions, Ch. 6.4 for the O notation, Ch. 8.5 for Taylor series.

1 MATHEMATICAL FUNCTIONS

1.1 The elementary functions of mathematics

Engineering mathematics draws heavily on the 'elementary functions' of mathematics of which there are surprisingly few. Basically, they comprise the polynomial functions, the trigonometric functions, and the logarithmic and exponential functions, all of which you will have met. There are also many 'special functions' defined for particular types of problem but we shall not be using any of these. For convenience some composite functions are also defined. In particular, you may not be familiar with the hyperbolic functions, which are defined as follows :

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

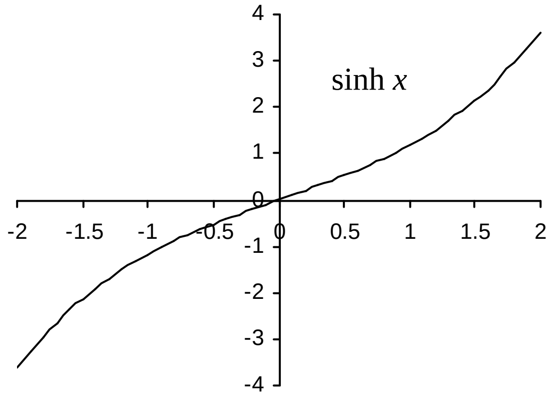
Not all functions can be represented analytically; indeed, any 'reasonable' curve you care to sketch on a graph of $f(x)$ versus x will define a function. For mathematical calculations, such functions may have to be represented numerically at discrete points. Alternatively, it may be possible to approximate a function over a certain range by using the elementary functions, perhaps in some combination. The range over which the approximation remains accurate may be quite large or quite small depending on how the original function behaves. This idea of approximating a complicated function with a simpler mathematical expression forms the basis for the following notes.

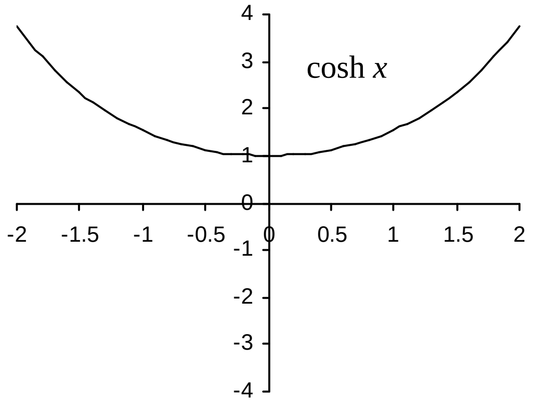
1.2 The symmetry of even and odd functions

We know that $x^2 = (-x)^2$ and $x^4 = (-x)^4$ and so on for all even powers of x. We also know that $x = -(-x)$ and $x^3 = -(-x)^3$ and so on for all odd powers of x. Thus, graphs involving even powers of x are symmetrical about the y-axis, and graphs involving odd powers of x are anti-symmetrical. This concept of symmetry is useful in more general cases and so we define :

> An *even* function $f(x)$ is defined to be one such that $f(-x) = f(x)$.
>
> An *odd* function $f(x)$ is defined to be one such that $f(-x) = -f(x)$.

It can be seen from the graphs below that $\sinh x$ is an odd function and $\cosh x$ is an even function of x.





1.3 How to sketch a function

Sketching the graph of a function can often be more useful than plotting it by computer. The interesting features of a function are usually the following :

- The values of $f(x)$ at salient points such as $x = -1, 0, 1$ and as $x \to \pm\infty$.
- Crossing points [*i.e.*, the values of x for which $f(x) = 0$].
- Maxima and minima [the values of x for which $f'(x) = 0$].
- Asymptotic behaviour [*e.g.*, the behaviour as x becomes very small or large].

For example, we sketch the function, $y = x^3 - 6x^2 + 11x - 6$

(i) At $x = 0$, $y = -6$

(ii) By inspection, we find that $x = 1$ is a root. Further examination then shows that the expression factorises into,

$$y = (x-1)(x-2)(x-3) \quad \rightarrow \quad \textbf{C}\text{rossing points where } y = 0 \text{ at } x = 1, 2 \text{ and } 3.$$

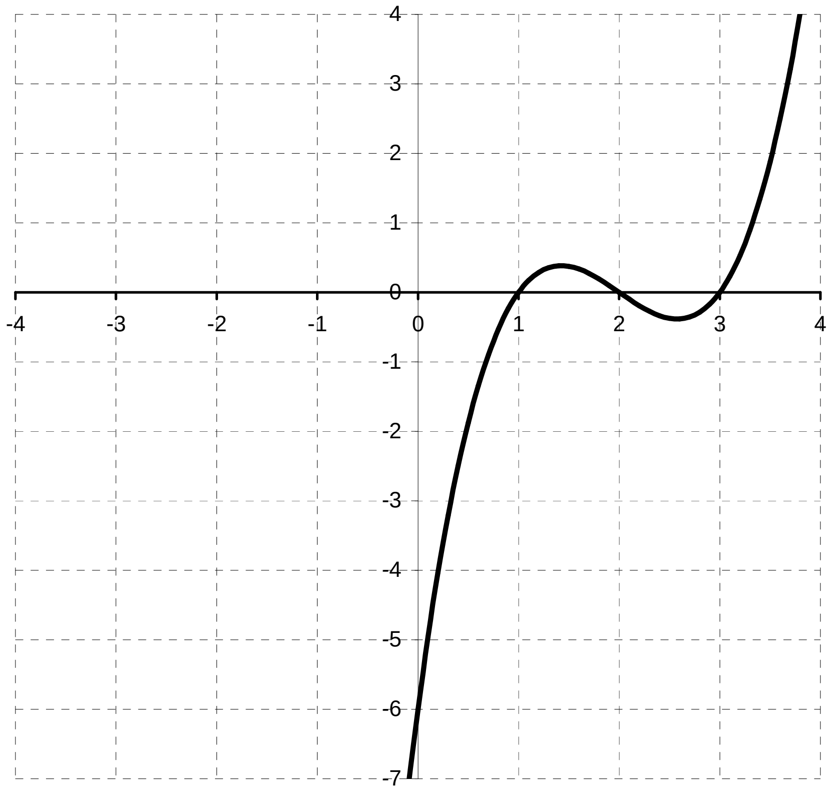
(iii) Differentiating, we obtain,

$$\frac{dy}{dx} = 3x^2 - 12x + 11 \quad \rightarrow \quad \text{Turning points at } \; x = \frac{12 \pm \sqrt{144-132}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

Thus, at $x = 2 - \frac{1}{\sqrt{3}} = 1.42$, $y = 0.38$, and at $x = 2 + \frac{1}{\sqrt{3}} = 2.58$, $y = -0.38$

(iv) As $x \to \pm\infty$, $y \to x^3$

We are now in a position to make a sketch of the function which actually looks like this :



2 APPROXIMATION BY SERIES

2.1 Taylor series

Taylor's theorem states that any function can be expanded as an infinite series around the point $x = a$ in the form,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

where $f'(a) = df/dx$ evaluated at $x = a$, etc. If we replace x by $(x + a)$, we obtain a series in x where x (which can be positive or negative) is now the deviation from a,

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \ldots$$

A special case is Maclaurin's series, obtained by putting $a = 0$ in either of the above forms,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

To derive Taylor's theorem, note that the objective is to find a power series of the form,

$$f(a+x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

where a_0, a_1, a_2, a_3, etc are constants to be determined. Setting $x = 0$ we find $a_0 = f(a)$. Successive differentiation then gives the other coefficients :

$$f'(x+a) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \quad \rightarrow \quad a_1 = f'(a)$$

$$f''(x+a) = 2a_2 + 3\times 2a_3 x + \ldots \quad \rightarrow \quad a_2 = f''(a)/2!$$

$$f'''(x+a) = 3\times 2a_3 + \ldots \quad \rightarrow \quad a_3 = f'''(a)/3!$$

Substituting back into the series then gives Taylor's expansion.

2.2 Power series expansions of the standard functions

The elementary functions of mathematics all have power series expansions which can be derived using Taylor's theorem and which can be found in the mathematics data book.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

2.3 The binomial expansion

The binomial expansion is found by extending the result of multiplying $(1 + x)$ by itself α times to situations where α is a rational number. The result, which can be found in the mathematics data book, is,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \ldots$$

The binomial expansion is particularly useful for approximating the reciprocal of a function. For example, if we wish to approximate the function $(1 + x^2)^{-1}$ when $|x| << 1$, we set $\alpha = -1$:

$$\frac{1}{(1+x^2)} = 1 - (x^2) + \frac{-1(-1-1)(x^2)^2}{2!} + \ldots = 1 - x^2 + x^4 + \ldots$$

2.4 The 'O' notation

The statement that a complicated function behaves like a simpler function in the vicinity of a particular value of x can be made more precise by use of the O notation. For example, we can replace the weak statement, $e^x \cong 1 + x$ for $|x| << 1$, by the stronger version,

$$e^x = 1 + x + O(x^2)$$

The notation $O(x^2)$ means 'a constant multiplied by x^2 plus terms involving higher powers of x'. Note that the 'approximately equals sign' has been replaced by an actual 'equals sign'.

The value of the O notation can be seen from the series for $(1 + x^2)^{-1}$ given above. Thus,

$$\frac{1}{(1+x^2)} = 1 - x^2 + O(x^4)$$

$$\frac{1}{(1+x^2)} = 1 - x^2 + x^4 + O(x^6)$$

The first statement informs us that there are no terms of order x^3 in the expansion as all neglected terms are of order x^4 and higher. The second statement is stronger as it informs us that, not only are there are no terms of order x^3, there are also no terms of order x^5.

2.5 Two useful tricks to obtain power series expansions

Change of variable

It is straightforward to use the standard expansions near $x = 0$ because terms involving the higher powers of x are simply neglected. However, in order to approximate a function in the vicinity of a point other than $x = 0$, it is first necessary to make a substitution.

Example : To approximate $\sin x$ near $x = \pi/4$, we let $y = x - \pi/4$. Then :

$$\sin x = \sin\left(y + \frac{\pi}{4}\right) = \sin(y)\cos\left(\frac{\pi}{4}\right) + \cos(y)\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\sin y + \cos y)$$

Expanding $\sin y$ and $\cos y$ around $y = 0$, we obtain,

$$\sin x = \frac{1}{\sqrt{2}}\left(y - \frac{y^3}{3!} + O(y^5) + 1 - \frac{y^2}{2!} + O(y^4)\right) = \frac{1}{\sqrt{2}}\left(1 + y - \frac{y^2}{2}\right) + O(y^3)$$

Approximation for large values of x

To approximate $f(x)$ for large x, try $y = 1/x$ and examine the behaviour as $y \to 0$.

Example : To approximate $\sin\left(\dfrac{1}{1+x}\right)$ as $x \to \infty$, we let $y = 1/x$. Then :

$$\sin\left(\frac{1}{1+x}\right) = \sin\left(\frac{y}{1+y}\right) = \left(\frac{y}{1+y}\right) - \frac{1}{3!}\left(\frac{y}{1+y}\right)^3 + O(y^5)$$

Expanding $(1+y)^{-1}$ and $(1+y)^{-3}$ using the binomial expansion, we obtain,

$$\sin\left(\frac{1}{1+x}\right) = y\,[1 - y + y^2 + O(y^3)] - \frac{y^3}{6}[1 - 3y + 6y^2 + O(y^3)] + O(y^5)$$

$$= y - y^2 + \frac{5}{6}y^3 + O(y^4)$$

This example shows how important it is to keep track of all contributions to the required order of the approximation. For example, we might naively have thought that $\sin[y/(1+y)]$ tends to $\sin y$ as $y \to 0$, but this is only true to the first order in y.

2.6 Two examples

Example 1 : Approximate $\cosh x$ near $x = 1$, accurate to terms of order $(x-1)^3$

Let $y = x - 1$. Then from the mathematics data book :

$$\cosh x = \cosh(1+y) = \cosh(1)\cosh(y) + \sinh(1)\sinh(y)$$

$$= \cosh(1)\left(1 + \frac{y^2}{2!} + O(y^4)\right) + \sinh(1)\left(y + \frac{y^3}{3!} + O(y^5)\right)$$

$$= \cosh(1) + \sinh(1)\,y + \frac{\cosh(1)}{2}y^2 + \frac{\sinh(1)}{6}y^3 + O(y^4)$$

where,

$$\cosh(1) = \frac{e + e^{-1}}{2}, \quad \sinh(1) = \frac{e - e^{-1}}{2}.$$

Example 2 : Approximate $(1+e^x)^{-1}$ near $x = 0$, accurate to terms of order x^2

From the mathematics data book :

$$(1+e^x) = 1 + \left(1 + x + \frac{x^2}{2} + O(x^3)\right)$$

Hence,

$$(1+e^x)^{-1} = \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + O(x^3)\right)^{-1}$$

Using the binomial expansion,

$$(1+e^x)^{-1} = \frac{1}{2}\left[1-\left(\frac{x}{2}+\frac{x^2}{4}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}+\frac{x^2}{4}\right)^2 + O(x^3)\right]$$

$$= \frac{1}{2}\left[1-\frac{x}{2}-\frac{x^2}{4}+\frac{x^2}{4} + O(x^3)\right]$$

$$= \frac{1}{2} - \frac{x}{4} + O(x^3)$$

3 INDETERMINATE FORMS

The function $\phi(x) = x/\tan(x)$ takes the value 0/0 when $x = 0$. This is an example of an *indeterminate form* meaning that it is impossible to establish its value in the usual way by substituting $x = 0$. If x is very small, even the output from a computer calculation may be unpredictable in such a situation. We now describe two ways to overcome this difficulty.

3.1 Power series expansion of the numerator and denominator

We suppose that $\phi(x) = f(x)/g(x)$ and that $f(0) = g(0) = 0$. Expanding the numerator and denominator as power series, we obtain,

$$\phi(x) = \frac{f(x)}{g(x)} = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...}$$

Where a_0, a_1, a_2 ... and b_0, b_1, b_2 ... are the constant coefficients in the series with $a_0 = f(0) = 0$ and $b_0 = g(0) = 0$. Cancelling a factor x we therefore obtain,

$$\phi(x) = \frac{a_1 + a_2 x + a_3 x^2 + ...}{b_1 + b_2 x + b_3 x^2 + ...} \rightarrow \frac{a_1}{b_1} \quad \text{as} \quad x \rightarrow 0$$

If, however, $a_1 = b_1 = 0$, then we cancel another x and find,

$$\phi(x) \rightarrow \frac{a_2}{b_2} \quad \text{as} \quad x \rightarrow 0$$

and so on.

3.2 l'Hôpital's rule

l'Hôpital's rule is a formal version of the analysis described above. Expanding the numerator and denominator as Taylor series about $x = 0$, we obtain,

$$\phi(x) = \frac{f(x)}{g(x)} = \frac{f(0) + xf'(0) + x^2 f''(0)/2! + x^3 f'''(0)/3! + ...}{g(0) + xg'(0) + x^2 g''(0)/2! + x^3 g'''(0)/3! + ...}$$

If $f(0) = g(0) = 0$, then,

$$\phi(x) \rightarrow \frac{f'(0)}{g'(0)} \quad \text{as} \quad x \rightarrow 0$$

If, further, $f'(0) = g'(0) = 0$, then,

$$\phi(x) \rightarrow \frac{f''(0)}{g''(0)} \quad \text{as} \quad x \rightarrow 0$$

and so on. l'Hôpital's rule is useful for simple cases but direct series expansion is usually preferable for more complicated examples.

3.3 Useful limits

The mathematics data book gives some useful limits of common indeterminate forms. Particularly important are the following two cases :

(i) As $x \rightarrow \infty$, $\frac{\ln x}{x^s} \rightarrow 0$ for any real value of $s > 0$. Thus, 'any power beats log'.

(ii) As $x \rightarrow \infty$, $x^s e^{-x} \rightarrow 0$ for any real value of s. Thus, 'exponential beats any power'.

3.4 A simple example : Find the value of $\phi(x) = x\,\mathrm{cotan}\,x$ as $x \to 0$.

As $x \to 0$, x cotan $x \to 0 \times \infty$ which is indeterminate. In order to get this into a recognisable 0/0 form, we rewrite the function as $\phi(x) = x/\tan x$.

For the numerator $f(x) = x$, $f'(x) = 1$, $f'(0) = 1$
For the denominator $g(x) = \tan x$, $g'(x) = \sec^2 x$, $g'(0) = 1$

Using l'Hôpital's rule,

$$\text{As } x \to 0,\ \ \phi(x) \ \to\ \frac{f'(0)}{g'(0)} \ = 1$$

3.5 A not so simple example : Find the value of $\phi(x) = x^2 \big/ [1 - x\,\mathrm{cotan}\,x]$ as $x \to 0$.

As $x \to 0$, x cotan x is indeterminate, and hence $\phi(x)$ itself is indeterminate. However, from the previous example we know that x cotan $x \to 1$ as $x \to 0$. Using this result we find,

$$\text{As } x \to 0,\ \ \phi(x) \ = \ \frac{x^2}{1 - x\,\mathrm{cotan}\,x} \ \to \ \frac{0}{1-1} \ = \ \frac{0}{0}$$

which is no help. We therefore settle down for a long haul and substitute cotan $x = \cos x/\sin x$ because it is easier to work with sin and cos :

$$\phi(x) \ = \ \frac{x^2 \sin x}{\sin x - x \cos x} \ = \ \frac{f(x)}{g(x)}$$

Numerator $f(x) = x^2 \sin x$ $\to$ $f(0) = 0$
Denominator $g(x) = \sin x - x \cos x$ $\to$ $g(0) = 0$

Numerator $f'(x) = 2x \sin x + x^2 \cos x$ $\to$ $f'(0) = 0$
Denomimator $g'(x) = \cos x - \cos x + x \sin x$ $\to$ $g'(0) = 0$

Numerator $f''(x) = 2 \sin x + 2x \cos x + 2x \cos x - x^2 \sin x$ $\to$ $f''(0) = 0$
Denominator $g''(x) = \sin x + x \cos x$ $\to$ $g''(0) = 0$

Numerator $f'''(x) = 2 \cos x + 4 \cos x - 4x \sin x - 2x \sin x - x^2 \cos x$ $\to$ $f'''(0) = 6$
Denominator $g'''(x) = \cos x + \cos x - x \sin x$ $\to$ $g'''(0) = 2$

Using l'Hôpital's rule,

$$\text{As } x \to 0,\ \ \phi(x) \ \to \ \frac{f'''(0)}{g'''(0)} = \frac{6}{2} = 3$$

We now see whether the job would have been any easier using direct series expansion. Thus :

$$f(x) \;=\; x^2 \sin x \;=\; x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathrm{O}(x^7) \right) \;=\; x^3 - \frac{x^5}{6} \;+\; \mathrm{O}(x^7)$$

$$g(x) \;=\; \sin x - x\cos x \;=\; \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathrm{O}(x^7) \right) - \; x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \mathrm{O}(x^6) \right)$$

$$= \; \frac{x^3}{3} - \frac{x^5}{30} \;+\; \mathrm{O}(x^7)$$

As $x \to 0$, $f(x) \;\to\; x^3 + \mathrm{O}(x^5)$, $g(x) \;\to\; \dfrac{x^3}{3} + \mathrm{O}(x^5)$, and hence $\phi(x) \to 3$